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Continuity Estimates for Ruin Probabilities

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A method of continuity analysis of ruin probabilities with respect to variation of parameters governing risk processes is proposed. It is based on the representation of the ruin probability as the stationary probability of a reversed process. We apply Kartashov's technique designed for continuity analysis of stationary distributions of general Markov chains in order to obtain desired continuity estimates. The method is illustrated by the Sparre Andersen and Markov modulated risk models. *Key words: Ruin probability, reversed process, general Markov chain, continuity estimate, stationary distribution, operator norm.*

1. INTRODUCTION

The probability of ruin is one of the basic characteristics of risk processes. It cannot, however, be found in an explicit form for many models of interest. Furthermore, parameters governing these models are often unknown and one can only give some bounds for their values. In such a situation continuity estimates become crucial.

In a general form, the continuity problem can be stated as follows. Let $R(t, \alpha)$ be a risk process governed by a parameter α. For example, if $R(t, \alpha)$ is the classical risk process (see Grandell [9] and Kalashnikov [11]), then one can take $\alpha = (\lambda, c, B)$, where λ is the intensity of the Poisson occurrence process, c is the gross premium rate, B is the distribution function (d.f.) of claim sizes. If $R(t, \alpha)$ is more complicated risk process, then one can define other appropriate parameters. Let us note that even for a relatively simple classical model the parameter takes values from a functional space. For a given process $R(t, \alpha)$, one can define a probability of ruin

$$\psi_\alpha(x) = \mathbf{P}\left(\inf_{t \geqslant 0} R(t; \alpha) < 0 \,\middle|\, R(0, \alpha) = x\right).$$

If $\mathbb{A}$ denotes the space of possible values of the parameter then one can view at the ruin probability as at a mapping ψ: $\mathbb{A} \to \Psi$, where Ψ is a functional space of all possible functions $\psi_\alpha(x)$, $x \geqslant 0$. Let us equip $\mathbb{A}$ and Ψ with metrics μ and v respectively. Then we can speak about continuity of the mapping ψ. Let us call the ruin probability ψ *continuous at point* α if

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$\mu(\alpha, \alpha') \to 0 \Rightarrow \nu(\psi_\alpha, \psi_{\alpha'}) \to 0.$

The metrics μ and ν should be convenient from computational point of view and reflect the core of the problem. For example, if the probability of ruin decays exponentially as in the Cramér case, then it is reasonable to take the distance in the form

$$\nu(\psi_\alpha, \psi_{\alpha'}) = \int_0^\infty e^{\varepsilon x} |\psi_\alpha - \psi_{\alpha'}| (dx) \tag{1.1}$$

for an appropriate constant $\varepsilon > 0$. If the probability of ruin decays like a power function (like it is in the presence of large claims with Pareto tails), then it is natural to consider a metric

$$\nu(\psi_\alpha, \psi_{\alpha'}) = \int_0^\infty x^\gamma |\psi_\alpha - \psi_{\alpha'}| (dx) \tag{1.2}$$

for an appropriate constant $\gamma > 0$. The same is true for the choice of metric μ. If we find an inequality

$$\nu(\psi_\alpha, \psi_{\alpha'}) \leqslant \phi(\mu(\alpha, \alpha')), \tag{1.3}$$

where the non-negative function ϕ is such that $\phi(0) = 0$ and $\phi(x) \to 0$ as $x \to 0$, then the inequality (1.3) is called a *continuity estimate* as it gives us the possibility to bound the desired quantity $\nu(\psi(\alpha), \psi(\alpha'))$ in terms of the distance $\mu(\alpha, \alpha')$.

The main purpose of this paper is to propose a general approach allowing us to obtain appropriate continuity estimates for ruin probabilities. We confine ourselves to the estimates of the distance (1.1). One can argue that, in practice, it is more interesting to estimate $|\psi_\alpha(u) - \psi_{\alpha'}(u)|$ for some (or, all) u. However, we can use the inequality

$$\sup_u e^{\varepsilon u} |\psi_\alpha(u) - \psi_{\alpha'}(u)| \leqslant \int_0^\infty e^{\varepsilon x} |\psi_\alpha - \psi_{\alpha'}| (dx) \tag{1.4}$$

in order to get the desired estimate:

$$|\psi_\alpha(u) - \psi_{\alpha'}(u)| \leqslant e^{-\varepsilon u} \nu(\psi_\alpha, \psi_{\alpha'})$$

for any $u \geqslant 0$. Note that the inequality (1.4) cannot be improved in the class of monotonically decreasing functions ψ_α and $\psi_{\alpha'}$.

This approach is based on the following three steps, each step being well-known in mathematical and actuarial literature. But it seems that their combination opens new possibilities.

The first step consists of identification of the ruin probability with a stationary probability for a specific random process which is called a *reversed process* (exact definitions will be given later on). Such identification is well-known and it has been investigated in recent works by Asmussen [1, 2] where the reader can also find further references. Let us denote this reversed process by V(t, α). The identification mentioned above means that

$$\psi_\alpha(x) = \lim_{t\to\infty} \mathbf{P}(V(t, \alpha) > x).$$

The second step consists of the embedding $V(t, \alpha)$ into a Markov process by equipping it with supplementary coordinates. So, instead of V, we consider an enlarged process

$$W(t, \alpha) = (V(t, \alpha), Z(t, \alpha)),$$

which is Markov. Such an embedding is widely used in queueing theory (see Cox [6], Gnedenko & Kovalenko [8]), complex systems theory (see Buslenko et al. [5]), control theory (see Davis [7]), and risk theory (see Schmidli [15]) in order to employ standard Markovain methods for the analysis of the underlying process. In these terms,

$$\psi_\alpha(x) = \pi_\alpha(\Gamma_x),$$

where $\pi_\alpha(\cdot)$ is stationary distribution of the Markov process $W(t, \alpha)$ and

$$\Gamma_x = \{W = (V, Z): V > x\}.$$

What is important, one can still regard the probability of ruin as the stationary distribution of a random process, but now this process is Markov for which the analysis of stationary distributions can be provided by standard methods.

The third step consists of the application of the continuity theory giving quantitative estimates of possible deviations of stationary distributions of the two Markov processes under the comparison governed by parameters α and α' respectively. More exactly, let $v(\pi_\alpha, \pi_{\alpha'})$ be an appropriate distance between stationary distributions of two Markov processes $W(t, \alpha)$ and $W(t, \alpha')$. And let $\mu(\alpha, \alpha')$ be an appropriate distance between α and α'. Then the continuity estimate means that the inequality of the form

$$v(\pi_\alpha, \pi_{\alpha'}) \leqslant \phi(\mu(\alpha, \alpha')) \tag{1.5}$$

holds for all $\alpha, \alpha' \in \mathbb{A}$, or may be, for α and α' belonging to a subset of $\mathbb{A}$, where function $\phi(x)$ is defined for $x \geqslant 0$, $\phi(0) = 0$, and ϕ is continuous at $x = 0$. This inequality is equivalent to (1.3).

Various methods can be used to obtain inequality (1.5). Let us list several relevant works in this direction: Borovkov [4], Kalashnikov [10, 11], Kartashov [13]. Further references can be found in these works. Here we shall use the results obtained by Kartashov [13]. They are stated for general Markov chains that is, they deal with the process $W(t, \alpha)$ having a discrete parameter $t = 0, 1, 2, \ldots$ That is why we consider only discrete time processes. But we start with usual continuous time risk processes successively reducing the problem to the discrete case.

We do not intent to develop a general theory in this paper but just illustrate how the combination of the three mentioned steps work in specific situations. We limit ourselves to only two examples. The first is the Sparre Andersen risk model, and here our result is actually a reformulation of a Kartashov's result (see [13]). It serves as a simple illustration of the approach where the second step (embedding into Markov process) is absent. The second example is a Markov modulated risk process where all steps are non-degenerated and the continuity estimates are new. The approach can be generalized to more complicated risk processes.

The paper is organized as follows. In Section 2 we construct a discrete time reversed process which allows us to define the ruin probability in terms of its stationary distribution.

All further constructions are illustrated by the two examples mentioned above. These examples have a double enumeration: Examples 3.1, 4.1, and 5.1 deal with Sparre Andersen model whereas Examples 3.2, 4.2, and 5.2 with the Markov modulated model.

Section 3 shows and illustrates how to embed the reversed process into a general Markov chain.

In Section 4 we introduce operators associated with general Markov chains and define norms of these operators and measures. These norms are used in order to obtain continuity estimates.

In Section 5, we state Kartashov's results concerning the continuity of stationary distributions of general Markov chains. These results are then applied to get continuity estimates for ruin probabilities in the S. Andersen and Markov modulated models (Theorems 1 and 2 respectively).

2. RUIN PROBABILITIES AND REVERSED PROCESSES

Let $R(t)$ be a risk process, about which we assume that $R(0)=x\geqslant 0$ is the initial capital, $\mathcal{T}^0=\{T_i^0\}$, $i\geqslant 0$, are successive occurrence times, $T_0^0=0$, and that paths of $R(t)$ are right-continuous. Then the probability of ruin can be defined as

$$\psi(x)=\mathbf{P}\left(\inf_{t\in\mathcal{T}^0} R(t)<0 \,\middle|\, R(0)=x\right).$$

For some reasons, we may want to consider a larger set of random times $\mathcal{T}\supset\mathcal{T}^0$. For example, if we consider a Markov modulated risk process, we might consider not only occurrence times but also the times of state changing of the modulating process. Let $\{T_i\}$ be the set of moments comprising $\mathcal{T}$, that is

$$\mathcal{T}=\{T_i\}_{i\geqslant 0},\quad T_0=0,$$

and let

$$R_n=R(T_n)$$

be the embedded risk process. As $\mathcal{T}\supset\mathcal{T}^0$ and the ruin can only occur at epoches from $\mathcal{T}^0$, we have

$$\psi(x)=\mathbf{P}\left(\inf_{n\geqslant 0} R_n<0 \,\middle|\, R_0=x\right). \tag{2.1}$$

Let

$$\xi_{n+1} = R_{n+1} - R_n. \tag{2.2}$$

Then,

$$R_n = x + \xi_1 + \cdots + \xi_n. \tag{2.3}$$

Hence,

$$\psi(x) = \mathbf{P}\left(\inf_{n \geqslant 1}(\xi_1 + \cdots + \xi_n) < -x\right).$$

Let

$$\psi(x, N) = \mathbf{P}\left(\inf_{1 \leqslant n \leqslant N}(\xi_1 + \cdots + \xi_n) < -x\right)$$

be the probability of ruin after N steps of the embedded process. For a fixed $N \geqslant 1$, define the process $\{V_n^{(N)}\}$ by the following recurrent equation

$$V_{n+1}^{(N)} = (V_n^{(N)} + \eta_{n+1}^{(N)})_+, \; 0 \leqslant n \leqslant N-1, \; V_0^{(N)} = 0, \tag{2.4}$$

where random variables $\eta_n^{(N)}$ will be defined later. Then

$$V_n^{(N)} = \max(0, \eta_n^{(N)}, \eta_n^{(N)} + \eta_{n-1}^{(N)}, \ldots, \eta_n^{(N)} + \cdots + \eta_1^{(N)}), \quad 1 \leqslant n \leqslant N. \tag{2.5}$$

Let us regard that both $\{\xi_n\}$ and $\{\eta_n^{(N)}\}$ are defined on the same probability space and thus, both $\{R_n\}$ and $\{V_n^{(N)}\}$ are also defined on the same probability space. Now, let us require that the sequence $\{\eta_n^{(N)}\}$ satisfies the conditions

$$\{V_n^{(N)} \leqslant R_{N-n}\} \Rightarrow \{V_n^{(N)} + \eta_{n+1}^{(N)} \leqslant R_{N-n} - \xi_{N-n}\}, \quad 0 \leqslant n \leqslant N-1, \tag{2.6}$$

$$\{V_n^{(N)} > R_{N-n}\} \Rightarrow \{V_n^{(N)} + \eta_{n+1}^{(N)} > R_{N-n} - \xi_{N-n}\}, \quad 0 \leqslant n \leqslant N-1. \tag{2.7}$$

It possible to put $\eta_{n+1}^{(N)} = -\xi_{N-n}$, which works in these arguments, but, in the case where ξ_k depend on the level of the risk process, we need in more general conditions (2.6) and (2.7). The reason for this is the fact that the governing random variables $\eta_n^{(N)}$ are supposed to be dependent of the values of the reversed process which do not coincide (in general) with the values of the risk process. An appropriate construction of variables η will be given below, after Assumption 1.

The following lemma is mostly algebraic rather than probabilistic and its proof uses arguments that are similar to those from works [2, 3]. However, there is some difference due to the fact that we consider the discrete time case and use the conditions (2.6) and (2.7).

LEMMA 1. *Given relations* (2.6) *and* (2.7),

$$\psi(x) = \lim_{N \to \infty} \mathbf{P}(V_n^{(N)} > x).$$

Proof. Let us fix $N > 0$ and assume that the ruin occurs within [1, N] that is, $\min_{1 \leqslant i \leqslant N} R_i < 0$. Let

$\varphi = \min\{i\colon R_i < 0\}.$

Then

$\psi(x, N) = \mathbf{P}(\varphi \leqslant N).$

By (2.7),

$$\{\varphi \leqslant N\} \subset \{R_{\varphi} < V^{(N)}_{N-\varphi}\} \subset \{R_{\varphi-1} < V^{(N)}_{N-\varphi+1}\} \subset \cdots \subset \{x = R_0 < V^{(N)}_N\}.$$

It follows that

$$\psi(x, N) \leqslant \mathbf{P}(V^{(N)}_N > x).$$

In the opposite case, there is no ruin within $[1, N]$ (perhaps, $\phi = \infty$), and hence, all $R_n \geqslant 0$, $0 \leqslant n \leqslant N$. But $V^{(N)}_0 = 0 \leqslant R_N$. Therefore

$$\begin{aligned}\{\varphi > N\} &\subset \{V^{(N)}_0 \leqslant R_N\} \cap \{\varphi > N\} \subset \{V^{(N)}_0 + \eta^{(N)}_1 \leqslant R_N - \xi_N\} \cap \{\varphi > N\}\\ &= \{V^{(N)}_0 + \eta^{(N)}_1 \leqslant R_{N-1}\} \cap \{\varphi > N\} = \{(V^{(N)}_0 + \eta^{(N)}_1)_+ \leqslant R_{N-1}\} \cap \{\varphi > N\}\\ &= \{V^{(N)}_1 \leqslant R_{N-1}\} \cap \{\varphi > N\} \subset \cdots \subset = \{V^{(N)}_n \leqslant R_0 = x\} \cap \{\varphi > N\}\\ &\subset \{V^{(N)}_n \leqslant R_0 = x\}\end{aligned}$$

which yields that

$$1 - \psi(x, N) \leqslant \mathbf{P}(V^{(N)}_N \leqslant x),$$

or

$$\mathbf{P}(V^{(N)}_N > x) \leqslant \psi(x, N).$$

Therefore

$$\psi(x, N) = \mathbf{P}(V^{(N)}_N > x),$$

which completes the proof. □

The statement of Lemma 1 is too general to be applied as the increments ξ_i may depend on the risk process. In order to make the construction tractable, let us impose the following restrictions which will be used in the rest of the paper.

Let $\{\sigma_n\}_{n \geqslant 1}$ be a stationary sequence which is regarded (without loss of generality) as a part of another stationary sequence $\{\sigma_n\}_{-\infty < n < \infty}$, taking values from an appropriate Polish space Σ, and

$$\xi_{n+1} = f(R_n, \sigma_{n+1}), \quad n \geqslant 0, \tag{2.8}$$

where $f\colon \mathbb{R} \times \Sigma \to \mathbb{R}$. Therefore,

$$R_{n+1} = R_n + f(R_n, \sigma_{n+1}). \tag{2.9}$$

Let us call sequence $\{\sigma_n\}$ governing.

ASSUMPTION 1.

$$g(r,\sigma)=r+f(r,\sigma) \tag{2.10}$$

is right-continuous with respect to r and has the following monotonicity property

$$\{r<r'\} \Rightarrow \{g(r,\sigma)<g(r',\sigma)\} \quad \forall\sigma\in\Sigma. \tag{2.11}$$

The condition (2.11) is natural. It infers that if an insurance company has a surplus r which is less than r' then, at the next step, its reserve $g(r,\sigma)$ will be still less than $g(r',\sigma)$.

For a fixed $\sigma\in\Sigma$ denote

$$g^{-1}(R,\sigma)=\inf\{r: g(r,\sigma)\geqslant R\}. \tag{2.12}$$

It follows that any R *from the set of values* of $g(r,\sigma)$ satisfy the equation

$$R=g^{-1}(R,\sigma)+f(g^{-1}(R,\sigma),\sigma)$$

or

$$R-f(g^{-1}(R,\sigma),\sigma)=g^{-1}(R,\sigma). \tag{2.13}$$

If, in particular, $f(r,\sigma)$ does not depend on r, then, evidently,

$$g^{-1}(R,\sigma)=R-f(\sigma).$$

For a fixed $N>0$, let

$$\eta_n^{(N)}=-f(g^{-1}(V_{n-1}^{(N)},\sigma_{N-n+1}),\sigma_{N-n+1})$$

and

$$V_{n+1}^{(N)}=(V_n^{(N)}+\eta_{n+1}^{(N)})_+,\quad V_0^{(N)}=0. \tag{2.14}$$

Let us check that, under Assumption 1, the conditions (2.6) and (2.7) hold. By (2.13), for any $n<N$,

$$V_n^{(N)}+\eta_{n+1}^{(N)}=V_n^{(N)}-f(g^{-1}(V_n^{(N)},\sigma_{N-n}),\sigma_{N-n})=g^{-1}(V_n^{(N)},\sigma_{N-n}).$$

If $V_n(N)\leqslant R_{N-n}$, then, by monotonicity of g^{-1},

$$g^{-1}(V_n^{(N)},\sigma_{N-n})\leqslant g^{-1}(R_{N-n},\sigma_{N-n})=R_{N-n-1}.$$

If $V_n(N)>R_{N-n}$, then

$$g^{-1}(V_n^{(N)},\sigma_{N-n})>g^{-1}(R_{N-n},\sigma_{N-n})=R_{N-n-1}.$$

Thus, (2.6) and (2.7) hold true and

$$\psi(x)=\lim_{N\to\infty}\mathbf{P}(V_N^{(N)}>x).$$

Let us return to definition (2.14) of the sequence $\{V_n^{(N)}\}$. Since $\{\sigma_n\}_{-\infty<n<\infty}$ is stationary, we can define another sequence $\{V_n\}_{n\geqslant 0}$ by the relation

$$V_{n+1}=(V_n+\eta_{n+1})_+,\quad V_0=0, \tag{2.15}$$

where

$$\eta_{n+1}=-f(g^{-1}(V_n,\sigma_{-n}),\sigma_{-n}). \tag{2.16}$$

Evidently, the following equality in distribution holds

$$\{V_n\}_{0\leqslant n\leqslant N}\stackrel{d}{=}\{V_n^{(N)}\}_{0\leqslant n\leqslant N},$$

and therefore,

$$\psi(x)=\lim_{n\to\infty}\mathbf{P}(V_n>x). \tag{2.17}$$

For brevity, the process $\{V_n\}$ will be referred to as a *reversed process*.

3. THE REVERSED PROCESS AND MARKOV CHAINS

In general, sequence $\{V_n\}$ is random with a complex correlation structure. In order to use a standard technique for its study, it is convenient to embed this sequence into a Markov chain (see Cox [6], Gnedenko & Kovalenko [8]).

ASSUMPTION 2. *Assume that the sequence* $\{V_n\}$ *defined by* (2.15) *can be embedded into a Markov chain*

$$W_n=(V_n,\tau_n), \tag{3.1}$$

where $\{\tau_n\}$ *is a sequence taking values from a Polish space* $\mathbb{T}$. *Thus*, W_n *takes values from* $\mathbb{W}=\mathbb{R}_+\times\mathbb{T}$.

Denote by

$$P(w,B)=\mathbf{P}(W_{n+1}=(V_{n+1},\tau_n)\in B\subset\mathbb{R}_+\times\mathbb{T}\mid W_n=w=(v,\tau))$$

the transition probability of chain (3.1) and by

$$\mathfrak{P}f(w)=\mathbf{E}(f(W_{n+1})\mid W_n=w) \tag{3.2}$$

its shift operator. Operator $\mathfrak{P}$ is defined for all functions f: $\mathbb{W}\to\mathbb{R}$ such that the right-hand side of (3.2) is finite.

EXAMPLE 3.1. Consider a S. Andersen model $R(t)$ (see Grandell [9] and Kalashnikov [12]). Let $R(0)=x$ and Z_n be successive i.i.d. claim sizes with the common distribution B and θ_n be successive i.i.d. inter-occurrence times with the common distribution A, $T_n^0=\theta_1+\cdots+\theta_n$, $n\geqslant 1$. Let $c>0$ be the gross premium rate. One can regard that this model is defined by the triple (A, c, B). Take $\mathscr{T}=\mathscr{T}^0$. Then

$$R_{n+1}=R_n+\xi_{n+1},\quad n\geqslant 0,$$

where

$$\xi_n = c\theta_n - Z_n.$$

Since $\{\xi_n\}$ are i.i.d., Assumption 2 holds if one takes $\sigma_n = \xi_n$ and $f(R, \sigma) = \sigma$. As $\{\xi_n\}$ consists of i.i.d.r.v's, it is a stationary sequence and the process $\{V_n\}$ can be defined by the equation

$$V_{n+1} = (V_n - \xi_{-n})_+, \quad V_0 = 0.$$

But $\xi_{-n} \stackrel{d}{=} \xi_n$. Therefore, one can consider the process

$$V_{n+1} = (V_n + Z_n - c\theta_n)_+, \tag{3.3}$$

where $Z_n - c\theta_n$ plays the role of η_{n+1}.

Process (3.3) is Markov and therefore, we do not need supplementary variables τ_n to embed it into a Markov chain. By (3.3), its shift operator has the form

$$\mathfrak{P}f(v) = \mathbf{E}f((v + Z - c\theta)_+),$$

where Z and θ are independent r.v's distributed as Z_n and θ_n respectively.

EXAMPLE 3.2. Let $R(t)$ $(R(0) = x))$ be the following Markov modulated risk process (see [2]). Assume that $J^0(t)$ is a continuous-time Markov process with finite state space $\mathbb{E} = \{1, 2,\dots, m\}$. The occurrence times from a Cox process of intensity $\lambda_{J^0(t)}$. If T_n^0 is the nth occurrence time and $J^0(T_n^0) = i$, then the corresponding claim size Z_n has the d.f. $B_i(u) = \mathbf{P}(Z_n \leqslant x)$ and does not depend on other characteristics of the process.

Denote by α_i the intensity with which $J^0(t)$ leaves state i that is, holding time of $J^0(t)$ at state i has the exponential distribution $1 - e^{\alpha_i u}$. upon leaving state i, $J^0(t)$ jumps to state $j \neq i$ with probability p_{ij}^0, $\Sigma_{j\neq i} p_{ij}^0 = 1$. Denote by c_i the gross premium rate at the state i. It is natural to view at the Markov modulated risk process as defined by the following parameter

$$\alpha = ((\lambda_i)_{i\in\mathbb{E}}, (\alpha_i)_{i\in\mathbb{E}}, (p_{ij}^0)_{i,j\in\mathbb{E}}, (c_i)_{i\in\mathbb{E}}, (B_i(u))_{i\in\mathbb{E}}). \tag{3.4}$$

We now consider the set $\mathscr{T} = \{T_n\}_{n\geqslant 0} \supset \mathscr{T}^0$ of all occurrence times and jumps of $J^0(t)$. For this, let us introduce another continuous-time Markov process $J(t)$ having the same state space $\mathbb{E}$ and the following parameters $\alpha_i + \lambda_i$—the intensity of the exponential holding time at state i,

$$p_{ij} = \frac{\alpha_i p_{ij}^0}{\alpha_i + \lambda_i}, \quad j \neq i, \tag{3.5}$$

$$p_{ii} = \frac{\lambda_i}{\alpha_i + \lambda_i}, \tag{3.6}$$

—the probabilities to jump $i \to j$, i, $j \in \mathbb{E}$.

Let I_n^0 be successive values of $J^0(t)$ (just after jumps) and I_n—successive values of $J(t)$ (also just after jumps). Evidently, I_n^0 is a homogeneous Markov chain with the stationary probabilities π_i^0, $i\in E$, satisfying the equations

$$\pi_j^0 = \sum_{i\in\mathbb{E}} \pi_i^0 p_{ij}^0, \quad j\in\mathbb{E}. \tag{3.7}$$

Similarly, I_n is also a homogeneous Markov chain having stationary probabilities

$$\pi_i = \beta\pi_i^0\left(1+\frac{\lambda_i}{\alpha_i}\right), \quad i\in\mathbb{E}, \tag{3.8}$$

where β is a norming coefficient,

$$\beta = \left(\sum_{i\in\mathbb{E}} \pi_i^0\left(1+\frac{\lambda_i}{\alpha_i}\right)\right)^{-1}. \tag{3.9}$$

Assume that the chain I_n is in the steady state that is, its initial distribution is the stationary one. Let $R_n = R(T_n)$. Then

$$R_{n+1} = R_n + \xi_{n+1},$$

where

$$\xi_{n+1} = c_{I_n}\theta_n^{(I_n)} - \delta_{I_n, I_{n+1}} Z_n^{(I_n)},$$

$\theta_n^{(i)} = T_{n+1} - T_n$ is the nth inter-occurrence time given $I_n = I(T_n) = i$ (therefore, it is exponentially distributed with the parameter $\alpha_i+\lambda_i$), δ_{ij} is the Kronecker delta, and $Z_n^{(i)}$ is the payoff made at time T_{n+1} (if it is an occurrence epoch) given that $I_n = i$ (that is, it is a r.v. having the d.f. B_i). Putting

$$\eta_{n+1}^{(N)} = -\xi_{N-n} = -c_{I_{N-n-1}}\theta_{N-n-1}^{(I_{N-n-1})} + \delta_{I_{N-n-1}, I_{N-n}} Z_{N-n-1}^{(I_{N-n-1})},$$

and using the stationarity of the sequence $\eta_i^{(N)}$, we define the following reversed process

$$V_{n+1} = (V_n + \delta_{I_{-n-1}, I_{-n}} Z_{-n-1}^{(I_{-n-1})} - c_{I_{-n-1}}\theta_{-n-1}^{(I_{-n-1})})_+, \quad n\geqslant 0. \tag{3.10}$$

The pair

$$W_n = (V_n, I_{-n}), \quad n\geqslant 0, \tag{3.11}$$

Forms a Markov chain taking values in $\mathbb{R}_+\times\mathbb{E}$. Let us write the shift operator $\mathfrak{P}$ for this chain. The transition probabilities of the chain $\{I_{-n}\}_{n\geqslant 0}$ are equal to

$$q_{ij} = \mathbf{P}(I_{-n-1} = j \mid I_{-n} = i) = \frac{\pi_j}{\pi_i} p_{ij}, \quad i,j\in\mathbb{E}. \tag{3.12}$$

Now

$$\mathfrak{P}f(v,i) = \mathbf{E}(f(V_{n+1}, I_{-n-1}) \mid V_n = v, I_{-n} = i).$$

Using (3.10) and (3.12), we obtain

$$\mathfrak{P}f(v,i) = p_{ii}\mathbf{E}f((v+Z^{(i)} - c_i\theta^{(i)})_+, i) + \frac{1}{\pi_i}\sum_{j\neq i}\pi_j p_{ij}\mathbf{E}f((v - c_j\theta^{(j)})_+, j). \tag{3.13}$$

4. OPERATORS ASSOCIATED WITH MARKOV CHAINS AND THEIR NORMS

Let W_n be a homogeneous Markov chain defined on the measurable space $(\mathbb{W}, \mathscr{W})$ with transition probability $P(w, \Gamma)$, $w \in \mathbb{W}$, $\Gamma \subset \mathscr{W}$, and stationary distribution π. Then one can associate the following two standard linear operators with the transition probability $P(w, \Gamma)$. The first one is the shift operator $\mathfrak{P}$ which operates on functions f: $\mathbb{W} \to \mathbb{R}_1$ by rule

$$\mathfrak{P}f(w) = \int_{\mathbb{W}} f(y)P(w, dy), \quad w \in \mathbb{W}. \tag{4.1}$$

The other operator (designated as $\mathfrak{P}^*$) operates on finite measures μ defined on $\mathscr{W}$ by the rule

$$\mathfrak{P}^*\mu(\Gamma) = \int_{\mathbb{W}} P(w, \Gamma)\mu(dw), \quad \Gamma \in \mathscr{W}, \tag{4.2}$$

(see [14]).

Actually, we have to compare probability distributions and transition probabilities of Markov chains which yields the necessity to consider measures taking both positive and negative values. Because of this, let us consider μ as an element of a space $\mathscr{C}$ of finite measures (not necessarily positive) on $(\mathbb{W}, \mathscr{W})$ and denote by $\mathscr{C}^+$ a positive cone in $\mathscr{C}$. The space $\mathscr{C}$ can be equipped with a norm $\|\cdot\|$ and therefore, can viewed as a Banach space. In this paper, we shall only consider the norm

$$\|\mu\|_\varphi = \int_{\mathbb{W}} \varphi(w)\, |\mu|\, (dw), \tag{4.3}$$

where φ is a positive measurable functions such that

$$0 < \kappa = \sup_{w \in \mathbb{W}} \frac{1}{\varphi(w)} < \infty \tag{4.4}$$

(cf. [13]).

Similarly, let us consider the kernel $P(w, \Gamma)$ in (4.1) and (4.2) as not necessarily a transition probability but as an element of $\mathscr{C}$ given w and as a measurable function of w provided that $\Gamma \in \mathscr{W}$ is fixed. Then one can define a norm of the operator $\mathfrak{P}^*$:

$$\|\mathfrak{P}^*\| = \sup(\|\mathfrak{P}^*\mu\| : \|\mu\| \leqslant 1). \tag{4.5}$$

In particular, if the norm of μ has the form (4.3), then

$$\|\mathfrak{P}^*\|_\varphi = \sup_{w \in \mathbb{W}} \frac{1}{\varphi(w)} \int \varphi(y)P(w, dy). \tag{4.6}$$

Note that the norm $\|\cdot\|_\varphi$ satisfies the following properties (see Kartashov [13])

$$\|\mu_1\| \leqslant \|\mu_1 + \mu_2\|, \quad \mu_1, \mu_2 \in \mathscr{C}^+, \tag{4.7}$$

$$\mu_1 \perp \mu_2,\ \mu_1, \mu_1 \in \mathscr{C}^+ \Rightarrow \|\mu_1 - \mu_2\| = \|\mu_1 + \mu_2\|, \tag{4.8}$$

$$|\mu(\mathbb{W})| \leqslant \kappa \|\mu\|, \quad \mu \in \mathscr{C}, \tag{4.9}$$

useful for more general constructions which are out of the scope of this paper.

Denoting by $\mathscr{F}$ the space of measurable functions on $\mathbb{W}$, one can equip it with the norm

$$\|f\|_\varphi = \sup_{w \in \mathbb{W}} \frac{|f(w)|}{\varphi(w)}, \quad f \in \mathscr{F}, \tag{4.10}$$

which induces the operator norm

$$\|\mathfrak{P}\|_\varphi = \sup\{\|\mathfrak{P}f\|_\varphi : \|f\|_\varphi \leqslant 1\} = \sup_{w \in \mathbb{W}} \frac{\mathfrak{P}\varphi(w)}{\varphi(w)}. \tag{4.11}$$

Note that

$$\|\mathfrak{P}\|_\varphi = \|\mathfrak{P}^*\|_\varphi \tag{4.12}$$

and

$$\|f\|_\varphi = \sup\left(\left|\int_{\mathbb{W}} f(w)\mu(dw)\right| : \|\mu\|_\varphi \leqslant 1\right).$$

In the remainder of the paper we limit ourselves to only Markov chains having the unique stationary probability and satisfying the following assumption.

ASSUMPTION 3. (Kartashov [13]) *For a Markov chain with the transition probability P and the unique stationary probability π, there exists a probability measure $G \in \mathscr{C}^+$ and a non-negative function $h \in \mathscr{F}$ such that*

(i) $\int_{\mathbb{W}} h(w)\pi(dw) > 0$, $\int_{\mathbb{W}} h(w)G(dw) > 0$;
(ii) the kernel $K(w, \Gamma) = P(w, \Gamma) - h(w)G(\Gamma)$ is non-negative;
(iii) $\|\mathfrak{K}\|_\varphi \leqslant \rho < 1$, where $\mathfrak{K}$ is the shift operator associated with the kernel K.

EXAMPLE 4.1. Let us return to the S. Andersen model. A transition kernel of the reversed process V_n has the form

$$P(v, \Gamma) = \mathbf{P}(v + Z_n - c\theta_n \in \Gamma, v + Z_n - c\theta_n > 0) + \mathbf{P}(v + Z_n - c\theta_n \leqslant 0)\delta_0(\Gamma), \tag{4.13}$$

where $\delta_0(\Gamma)$ is the probability measure concentrate at 0.

Let us assume that there exists an $\varepsilon > 0$ such that

$$\mathbf{E}\exp(\varepsilon(Z_n - c\theta_n)) = \rho < 1. \tag{4.14}$$

In particular, condition (4.14) yields the positivity of the safety loading which, in turn, guarantees that the reversed process $\{V_n\}$ has the limiting distribution.

We shall consider the norm $\|\cdot\|_\varphi$, where $\varphi(v) = e^{\varepsilon v}$. For this, let us split the transition probability of Markov chain V_n (cf. [13]):

$P(v; \Gamma) = K(v, \Gamma) + h(v)\delta_0(\Gamma),$

where $h(v) = \mathbf{P}(v + Z_n - c\theta_n \leqslant 0)$. By (4.13), $K(v, \Gamma) \geqslant 0$. We will show that the norm of $\mathfrak{K}$ is less than or equal to ρ. Actually,

$$\begin{aligned} \|\mathfrak{K}\|_\varphi &= \sup_{v \geqslant 0} e^{-\varepsilon v} \int_0^\infty e^{\varepsilon y} K(v; dy) \\ &= \sup_{v \geqslant 0} e^{-\varepsilon v}\, \mathbf{E}(e^{\varepsilon(v + Z_n - c\theta_n)}; Z_n - c\theta_n + v > 0) \\ &\leqslant \mathbf{E}\, e^{\varepsilon(Z_n - c\theta_n)} \equiv \rho < 1. \end{aligned} \tag{4.15}$$

Inequality (4.15) will be used for continuity estimate in the following section.

EXAMPLE 4.2. Let us turn to the Markov modulated risk process. Denote

$$b_i(r) = \mathbf{E}\exp(rZ^{(i)}) \tag{4.16}$$

and assume that $b_i(r) < \infty$ for some $r > 0$ and all $i \in \mathbb{E}$. We have also, for $r > 0$,

$$\mathbf{E}\exp(-r\theta^{(i)}) = \frac{\alpha_i + \lambda_i}{\alpha_i + \lambda_i + rc_i}. \tag{4.17}$$

Put

$$p_{ij}(r) = \begin{cases} p_{ii}\mathbf{E}\, e^{r(Z^{(i)} - c_i\theta^{(i)})} = \lambda_i b_i(r)/(\alpha_i + \lambda_i + c_i r), & j = i \\ p_{ij}\mathbf{E}\, e^{rc_i\theta^{(i)}} = \alpha_i p^0_{ij}/(\alpha_i + \lambda_i + c_i r), & j \neq i, \end{cases}$$

where probabilities p_{ij} are defined in (3.5) and (3.6). denote by $\Pi(r)$ the matrix with elements $p_{ij}(r)$. This matrix is positive and therefore, its spectral radius **Spr** $\Pi(r)$ is equal to maximal eigenvalue $d(r)$ which is positive. Note that $d(0) = 1$. Define

$$\varepsilon^* \equiv \sup\{r\colon\ \mathbf{Spr}\,\Pi(r) < 1\}. \tag{4.18}$$

The constant ε^* is associated with the Cramér condition, cf. Grandell [9, § 4.4].

ASSUMPTION 4. *Throughout the remainder of the example, we assume that* $\varepsilon^* > 0$ *and that* ε $(0 < \varepsilon \leqslant \varepsilon^*)$ *satisfies the inequality*

$\mathbf{Spr}\,\Pi(\varepsilon) < 1.$

In particular, if we deal with the "usual case" **Spr** $\Pi(\varepsilon^*) = 1$, then ε should be less than ε^*. But if **Spr** $\Pi(\varepsilon^*) < 1$, then it is possible (and desirable) to choose $\varepsilon = \varepsilon^*$.

Let us introduce another matrix $Q(r)$ with elements

$$q_{ij}(r) = \begin{cases} q_{ii}\mathbf{E}\, e^{r(Z^{(i)} - c_i\theta^{(i)})}, & j = 1 \\ q_{ij}\mathbf{E}\, e^{-rc_j\theta^{(i)}}, & j \neq i, \end{cases}$$

where probabilities q_{ij} are defined in (3.12). Matrix $Q(r)$ is associated with reversed process. Evidently,

$$Q(r) = T^{-1}\Pi^t(r)T,$$

where T is a diagonal matrix with diagonal elements π_i, T^{-1} is the inverse of T, and $\Pi^t(r)$ is the transposition of $\Pi(r)$. Therefore, the maximal eigenvalue of $Q(r)$ is equal to $d(r)$. Let us denote by $\gamma(r)$ the eigenvalue (column) of the matrix $Q(r)$ corresponding to the eigenvalue $d(r)$ and by $\gamma^{\Pi}(r)$ the corresponding eigenvector of $\Pi^t(r)$. Then

$$\gamma(r) = T^{-1}\gamma^{\Pi}(r).$$

Denote by $\gamma_i(r)$ components of the vector $\gamma(r)$; they all are positive by the Perron–Frobenius theory. For definiteness, let us consider that $\gamma_1(r) = 1$. Evidently, $\gamma(r)$ is a continuous function of r.

Now, let us return to the reversed process W_n (see (3.10) and (3.11)) and denote its transition probability by $P((v, i); (\Gamma, j))$, where Γ is a Borel set belonging to $[0, \infty)$. The shift operator $\mathfrak{P}$ defining this transition probability is given in (3.13).

Let us split the transition probability P in the following way

$$P((v, i); (\Gamma, j)) = K((v, i); (\Gamma, j)) + h(v, i)G(\Gamma, j), \tag{4.19}$$

where

$$h(v, i) = h_i^0 \min_{i \in \mathbb{E}} \mathbf{P}(v + Z^{(i)} - c_i\theta^{(i)} \leqslant 0),$$

$$h_i^0 = \frac{1}{\pi_i} \min_{j \in \mathbb{E}} p_{ji},$$

$$G(\Gamma, j) = \delta_0(\Gamma)\pi_j.$$

Now we prove that kernel K satisfies Assumption 3. Evidently, Assumptions 3(i, ii) hold. The rest of the example is devoted to the proof that Assumption 3(iii) is also true.

Denote

$$\varphi_r(v, i) = \gamma_i(r)\, e^{rv}, \quad r \geqslant 0, \ v \geqslant 0, \quad i \in \mathbb{E}. \tag{4.20}$$

LEMMA 2. *Let Assumption* 4 *hold. Then there exists a constant* $v^* = v^*(\varepsilon) \geqslant 0$ *such that*

$$\sup_{v \geqslant v^*} \max_{i \in \mathbb{E}} \frac{\mathfrak{K}\varphi_\varepsilon(v, i)}{\varphi_\varepsilon(v, i)} = \rho_1(\varepsilon) < 1. \tag{4.21}$$

Proof. Since $K \leqslant P$, it suffices to prove that

$$\sup_{v \geqslant v^*} \max_{i \in \mathbb{E}} \frac{\mathfrak{P}\varphi_\varepsilon(v, i)}{\varphi_\varepsilon(v, i)} = \rho_1(\varepsilon) < 1. \tag{4.22}$$

Using (3.13), we have

$$\gamma_i(\varepsilon)\int_0^\infty e^{\varepsilon y}P((v,i);(dy,i)) = q_{ii}\gamma_i(\varepsilon)\mathbf{E}\exp(\varepsilon(v+Z^{(i)}-c_i\theta^{(i)})_+)$$
$$\leqslant q_{ii}\gamma_i(\varepsilon)\,(\mathbf{E}\exp(\varepsilon(v+Z^{(i)}-c_i\theta^{(i)}))+1)$$
$$= q_{ii}(\varepsilon)\gamma_i(\varepsilon)\,e^{\varepsilon v}+q_{ii}\gamma_i(\varepsilon). \qquad (4.23)$$

Similarly, if $j\neq i$,

$$\gamma_j(\varepsilon)\int_0^\infty e^{\varepsilon y}P((v,i);(dy,j)) \leqslant q_{ij}\gamma_j(\mathbf{E}\exp(\varepsilon(v-c_j\theta^{(j)}))+1)$$
$$= q_{ij}(\varepsilon)\gamma_j(\varepsilon)\,e^{\varepsilon v}+q_{ij}\gamma_j(\varepsilon). \qquad (4.24)$$

Taking into account that

$$\sum_{j\in\mathbb{E}} q_{ij}(\varepsilon)\gamma_j(\varepsilon)=d(\varepsilon)\gamma_i(\varepsilon), \quad i\in\mathbb{E},$$

and

$$\sum_{j\in\mathbb{E}} q_{ij}=1,$$

we have

$$\frac{\mathfrak{P}\varphi_\varepsilon(v,i)}{\varphi_\varepsilon(v,i)}=\frac{1}{\gamma_i(\varepsilon)\,e^{\varepsilon v}}\sum_{j\in\mathbb{E}}\gamma_j(\varepsilon)\int_0^\infty e^{\varepsilon y}P((v,i);(dy,j))\leqslant d(\varepsilon)+D(\varepsilon)\,e^{-\varepsilon v},$$

where

$$D(\varepsilon)=\frac{\max_{i\in\mathbb{E}}\gamma_i(\varepsilon)}{\min_{i\in\mathbb{E}}\gamma_i(\varepsilon)}.$$

Taking

$$v^* = -\frac{1}{\varepsilon}\ln\frac{1-d(\varepsilon)}{2D(\varepsilon)},$$

we arrive at the inequality (4.22) with

$$\rho_1(\varepsilon)=\frac{1+d(\varepsilon)}{2}<1,$$

which completes the proof. □

Remark 1. It follows from the proof of Lemma 2 that

$$\rho_1[\varepsilon_1,\varepsilon_2]\equiv\sup_{r\in[\varepsilon_1,\varepsilon_2]}\rho_1(r)<1 \qquad (4.25)$$

for any $0<\varepsilon_1\leqslant\varepsilon_2\leqslant\varepsilon$.

Furthermore, for all $r\in[\varepsilon_1,\varepsilon_2]$, there exists the same constant v^* satisfying Lemma 2.

LEMMA 3. *Let Assumption 4 hold. Then there exists* $0<\varepsilon_*\leqslant\varepsilon^*$ *such that, for any* $v^*\geqslant 0$,

$$\sup_{v \leqslant v^*} \max_{i \in \mathbb{E}} \frac{\mathfrak{K}\varphi_{\varepsilon_*}(v, i)}{\varphi_{\varepsilon_*}(v, i)} = \rho_2(v^*) < 1. \tag{4.26}$$

Proof. Similarly to (4.23) and (4.24), we have

$$\frac{\mathfrak{P}\varphi_0(v, i)}{\varphi_0(v, i)} = \sum_{j \in \mathbb{E}} P((v, i); ([0, \infty), j)) = d(0) = 1,$$

where $\varphi_0(v, i) = \gamma_i(0) \equiv 1$, and (4.19) yields that

$$\frac{\mathfrak{K}\varphi_0(v, i)}{\varphi_0(v, i)} = 1 - h_i^0 \sum_{j \in \mathbb{E}} \pi_j \min_{i \in \mathbb{E}} \mathbf{P}(v + Z^{(i)} - c_i\theta^{(i)} \leqslant 0).$$

Denote

$$s(v) = \min_{i \in \mathbb{E}} \mathbf{P}(v + Z^{(i)} - c_i\theta^{(i)} \leqslant 0).$$

Evidently, $s(v) > 0$ for any $v \geqslant 0$, and $s(v) \to 0$ as $v \to \infty$.

It follows that (4.26) holds for $\varepsilon_* = 0$ and the left hand side of (4.26) does not exceed

$$\rho_0^2(v^*) = 1 - s(v^*) \min_{i,j \in \mathbb{E}} \frac{p_{ij}}{\pi_i}.$$

The continuity of $\gamma_i(r)$ with respect to r and Assumption 4 infer that relation (4.26) holds for a positive (sufficiently small) ε_* and appropriate $\rho_2(v^*) < 1$, which completes the proof. □

Now, let $\varepsilon > 0$ satisfy Assumption 4 and $\varepsilon^* \leqslant \varepsilon$ satisfy Lemma 3. Put

$$\varphi(v, i) = \gamma_i(r(v)) \exp(r(v)v), \quad v \geqslant 0, i \in \mathbb{E}, \tag{4.27}$$

where

$$r(v) = \varepsilon_* + \frac{(\varepsilon - \varepsilon_*)\chi v}{1 + \chi v}, \quad \chi > 0. \tag{4.28}$$

Evidently, $r(0) = \varepsilon_*$ and $r(v) \to \varepsilon$ as $v \to \infty$. Note also that the function $r(v)v$ satisfies the Lipschitz condition

$$|r(v_1)v_1 - r(v_2)v_2| \leqslant \varepsilon |v_1 - v_2|. \tag{4.29}$$

Furthermore,

$$\|G\|_\varphi = \sum_{i \in \mathbb{E}} \pi_i \gamma_i(\varepsilon_*)$$

(see (4.19)) and

$$\kappa \leqslant \kappa[\varepsilon_*, \varepsilon] = \max_{i \in \mathbb{E}} \frac{1}{\gamma_i^*},$$

where

$$\gamma_i^* = \inf_{r \in [\varepsilon_*, \varepsilon]} \gamma_i(r).$$

LEMMA 4. *Let Assumption* 4 *hold. If* φ *is defined by* (4.27) *and* (4.28), *then* $\|\mathfrak{K}\|_\varphi = \rho < 1$ *for sufficiently small* $\chi > 0$.

Proof. Let

$$f_v(z, i) = \gamma_i(r(v)) \exp(r(v)z), \quad i \in \mathbb{E}, z \geqslant 0, v \geqslant 0.$$

Note that

$$f_v(v, i) = \varphi(v, i).$$

It follows, from the continuity of $\gamma_i(r)$, the Lipschitz condition (4.29), and Assumption 4, that, for any $\delta > 0$, there exists $\chi_1 > 0$ such that

$$\left| \frac{\mathfrak{K} f_v(v, i)}{f_v(v, i)} - \frac{\mathfrak{K}\varphi(v, i)}{\varphi(v, i)} \right| \leqslant \delta$$

for all $\chi \leqslant \chi_1$, $i \in \mathbb{E}$, and $v \geqslant 0$. Therefore,

$$\left| \|\mathfrak{K}\|_\varphi - \sup_{i \in \mathbb{E}, v \geqslant 0} \frac{\mathfrak{K} f_v(v, i)}{f_v(v, i)} \right| \leqslant \delta.$$

But, by Lemma 2 and Remark 1, there exist $v^* \geqslant 0$ and $0 < \rho_1 < 1$ such that

$$\sup_{i \in \mathbb{E}, v \geqslant v^*} \frac{\mathfrak{K} f_v(v, i)}{f_v(v, i)} \leqslant \rho_1.$$

By Lemma 3, there exist $0 < \rho_2(v^*) < 1$ and $\chi_2 > 0$ such that, for all $\chi < \chi_2$,

$$\sup_{i \in \mathbb{E}, v \geqslant v^*} \frac{\mathfrak{K} f_v(v, i)}{f_v(v, i)} \leqslant \rho_2(v^*).$$

This yields that

$$\|\mathfrak{K}\|_\varphi \leqslant \max(\rho_1, \rho_2(v^*)) + \delta$$

for all $\chi \leqslant \min(\chi_1, \chi_2)$. Since δ can be chosen as small as necessary, the lemma is proved. □

5. CONTINUITY ESTIMATES

Let us state the following result proved in Kartashov [13, Theorem 8].

LEMMA 5. *Let Assumption* 3 *hold and* $\|\mathfrak{P}\|_\varphi < \infty$. *Then each Markov chain with the transition probability* P' *corresponding to the shift operator* $\mathfrak{P}'$ *and satisfying*

$$\Delta \equiv \|\mathfrak{P}' - \mathfrak{P}\|_\varphi < \frac{1-\rho}{1 + \|\pi\|_\varphi \kappa\rho} \equiv \Delta_0, \tag{5.1}$$

has a unique invariant probability measure π' *and*

$$\|\pi' - \pi\|_{\varphi} \leqslant \frac{\Delta\|\pi\|_{\varphi}}{\Delta_0 - \Delta}, \tag{5.2}$$

where the norm $\|\pi\|_{\varphi}$ *can be estimated as*

$$\|\pi\|_{\varphi} \leqslant \frac{\|G\|_{\varphi}}{1-\rho}. \tag{5.3}$$

Lemma 5 serves as a source for desired continuity estimates in our paper. We shall use its assertion the following simpified form.

COROLLARY 1. *Under the assumptions and notation of Lemma* 5,

$$\|\pi' - \pi\|_{\varphi} \leqslant \frac{\Delta\|G\|_{\varphi}}{(1-\rho)(\Delta_0 - \Delta)}, \tag{5.4}$$

if

$$\Delta < \frac{(1-\rho)^2}{1 + (\kappa\|G\|_{\varphi} - 1)_{\rho}} \equiv \Delta_0. \tag{5.5}$$

EXAMPLE 5.1. Let us finish considering the S. Andersen risk process under Assumption (4.14). The process is completely determined by the triple $\alpha = (A, c, B)$ (see Example 3.1). As we have seen, the probability $\psi_{\alpha}(v)$ of ruin coincides with the stationary probability for the Markov chain $\{V_n\}$ (see (3.3)) to exceed the level v. Let $\alpha' = (A', c', B')$ be the triple, defining another risk process, its ruin probability being $\psi_{\alpha'}(v)$. Let ε be the constant satisfying (4.14) and $\varphi(v) = e^{\varepsilon v}$ (like in Example 4.1). Then the distance $v(\psi_{\alpha}, \psi_{\alpha'})$ between corresponding ruin probabilities can be expressed in terms of the stationary distributions π and π' as follows:

$$v(\psi_{\alpha}, \psi_{\alpha'}) = \int e^{\varepsilon x}|\psi_{\alpha} - \psi_{\alpha'}|(dx) = \|\pi - \pi'\|_{\varphi}. \tag{5.6}$$

Denote

$$\mu(\alpha, \alpha') = \|\mathfrak{P} - \mathfrak{P}'\|_{\varphi} \tag{5.7}$$

and

$$A_c(y) = \mathbf{P}(c\theta \leqslant y) = A(y/c),$$

$$A'_{c'}(y) = \mathbf{P}(c'\theta' \leqslant y) = A'(y/c').$$

Then, by definition,

$$\mu(\alpha, \alpha') = \sup_{v \geqslant 0} e^{-\varepsilon v} \int_0^{\infty} e^{\varepsilon y}|P'(v; dy) - P(v; dy)|$$

$$\leqslant \sup_{v \geqslant 0} e^{-\varepsilon v}\left(\int_0^\infty e^{\varepsilon(v+z)}|B' - B|(dz)\right.$$

$$\left. + \int_0^\infty e^{\varepsilon(v+z)} B(dz) \int_0^\infty |A'_{c'} - A_c|(dy)\right)$$

$$= \mathbf{E}\, e^{\varepsilon Z} \int_0^\infty |A'_{c'} - A_c|(dy) + \int_0^\infty e^{\varepsilon z}|B' - B|(dz). \tag{5.8}$$

THEOREM 1. *If the S. Andersen model satisfies condition* (4.14), $\mu(\alpha, \alpha')$ *is defined by* (5.7), *and* $v(\psi_\alpha, \psi_{\alpha'})$ *by* (5.6), *then, for* $\mu(\alpha, \alpha') \leqslant (1-\rho)^2$,

$$v(\psi_\alpha, \psi_{\alpha'}) \leqslant \frac{\mu(\alpha, \alpha')}{(1-\rho)((1-\rho)^2 - \mu(\alpha, \alpha'))}, \tag{5.9}$$

where ρ *is taken from* (4.14).

Proof. In Example 4.1, it was shown that $\kappa = \|G\|_\varphi = 1$. Plugging these expressions into Corollary 1, we arrive at (5.9) □.

Remark 2. One can replace $\mu(\alpha, \alpha')$ in the continuity estimate (5.9) by its upper bound (5.8), or any other appropriate bound expressed in terms of the triples α and α'.

EXAMPLE 5.2. We derive now a continuity estimate for the Markov modulated risk process which is governed by parameter α defined in (3.4). Let all assumptions of Lemma 4 hold and function $\varphi(v, i)$ be defined by (4.27) and (4.28).

First of all, using the expressions for $\|G\|_\varphi$ and κ listed in Example 4.2, and Corollary 5.4, we arrive at the following general inequality

$$\|\pi' - \pi\|_\varphi \leqslant \frac{\|\mathfrak{P}' - \mathfrak{P}\|_\varphi}{(1-\rho)\,(\Delta_0 - \|\mathfrak{P}' - \mathfrak{P}\|_\varphi)} \sum_{j \in \mathbb{E}} \pi_j \gamma_j(\varepsilon_*), \tag{5.10}$$

where $0 < \rho < 1$ is taken from Lemma 4,

$$\Delta_0 = \frac{(1-\rho)^2}{1 + (B+1)\rho}, \tag{5.11}$$

$$B = \max_{i,j \in \mathbb{E}, r \in [\varepsilon_*, \varepsilon]} \frac{\gamma_j(\varepsilon_*)}{\gamma_i(r)}. \tag{5.12}$$

The estimate (5.10) is valid if $\|\mathfrak{P}' - \mathfrak{P}\|_\varphi \leqslant \Delta_0$.

Now, let us estimate quantities $\|\pi' - \pi\|_\varphi$ and $\|\mathfrak{P}' - \mathfrak{P}\|_\varphi$ in terms of the risk process. We start with $\|\pi' - \pi\|_\varphi$. Evidently, the probability of ruin can be presented in the form

$$\psi(x) = \sum_{i \in \mathbb{E}} \int_x^\infty \pi(dv, i).$$

Denote

$$v(\psi_\alpha, \psi'_\alpha) = \int_0^\infty e^{\varepsilon v} |\psi_\alpha - \psi'_\alpha| (dv). \tag{5.13}$$

By (4.28),

$$\varepsilon v \leqslant \frac{\varepsilon - \varepsilon_*}{\chi} + r(v)v. \tag{5.14}$$

Actually, (5.14) is equivalent to the inequality

$$\varepsilon v \chi \leqslant \varepsilon - \varepsilon_* + r(v)v\chi, \tag{5.15}$$

And (4.28) can be written as

$$r(v)v\chi = \varepsilon_* + \varepsilon v \chi - r(v).$$

Therefore, (5.15) is equivalent to $\varepsilon - r(v) \geqslant 0$ that is, of course, true.

It follows that

$$\begin{aligned} v(\psi_\alpha, \psi'_\alpha) &\leqslant e^{(\varepsilon - \varepsilon_*)/\chi} \int_0^\infty e^{r(v)v} |\psi_\alpha - \psi'_\alpha| (dv) \\ &\leqslant e^{(\varepsilon - \varepsilon_*)/\chi} \sum_{i \in \mathbb{E}} \int_0^\infty e^{r(v)v} |\pi'(dv, i) - \pi(dv, i)| \\ &\leqslant \kappa[\varepsilon_*, \varepsilon]\, e^{(\varepsilon - \varepsilon_*)/\chi} \|\pi' - \pi\|_\varphi. \end{aligned}$$

Now, let

$$\begin{aligned} \mu(\alpha, \alpha') &= \|\mathfrak{P}' - \mathfrak{P}\|_\varphi \\ &= \sup_{v \geqslant 0} \max_{i \in \mathbb{E}} \sum_{j \in \mathbb{E}} \int_0^\infty \frac{\gamma_j(r(y))\, e^{r(y)y}}{\gamma_i(r(v))\, e^{r(v)v}} |P'((v, i); (dy, j)) - P((v, i); (dy, j))| \\ &\leqslant C \sup_{v \geqslant 0} \max_{i \in \mathbb{E}} \sum_{j \in \mathbb{E}} \int_0^\infty e^{r(y)y - r(v)v} |P'((v, i); (dy, j)) - P((v, i); (dy, j))|, \end{aligned}$$

where

$$C = \sup_{r_1, r_2 \in [\varepsilon_*, \varepsilon], i, j \in \mathbb{E}} \frac{\gamma_j(r_1)}{\gamma_i(r_2)}.$$

Let us denote, for all $i \in \mathbb{E}$,

$$A_{1i}(y) = P(c_i \theta^{(i)} \leqslant y) = A_i(y/c_i),$$

$$A'_{1i}(y) = P(c'_i \theta^{(i)'} \leqslant y) = A_i(y/c'_i).$$

Then, similarly to (5.8), by the Lipschitz condition (4.29),

$$\sup_{v \geqslant 0} \int_0^\infty e^{r(y)y - r(v)v} |P'((v, i); (dy, i)) - P((v, i); (dy, i))|$$

$$\leqslant b_i(\varepsilon)\int_0^\infty |A'_{1i}-A_{1i}|(dy)+\int_0^\infty e^{\varepsilon z}|B'_i-B_i|(dz),$$

where $b_i(\varepsilon)$ is defined in (4.16). Similarly, for $j\neq i$,

$$\sup_{v\geqslant 0}\int_0^\infty e^{r(y)y-r(v)v}|P'((v,i);(dy,j))-P((v,i);(dy,j))|$$

$$\leqslant \int_0^\infty |A'_{1j}-A_{1j}|(dy)+\int_0^\infty |B'_j-B_j|(dz).$$

This follows that

$$\mu(\alpha,\alpha')\leqslant \max_{i\in\mathbb{E}}\left(b_i(\varepsilon)\int_0^\infty |A'_{1i}-A_{1i}|(dy)+\int_0^\infty e^{\varepsilon z}|B'_i-B_i|(dz)\right.$$
$$\left.+\sum_{j\neq i}\int_0^\infty (|A'_{1j}-A_{1j}|+|B'_j-B_j|)(dy)\right). \tag{5.16}$$

THEOREM 2. *If the Markov modulated risk process satisfies Assumption* 4 *and* $\varepsilon^*>0$ *is taken from Lemma* 3, *then the following continuity estimate holds*

$$v(\psi_\alpha,\psi_{\alpha'})\leqslant\frac{\kappa[\varepsilon_*,\varepsilon]\,e^{(\varepsilon-\varepsilon_*)/\chi}\mu(\alpha,\alpha')}{(1-\rho)(\Delta_0-\mu(\alpha,\alpha'))}\sum_{j\in\mathbb{E}}\pi_j\gamma_j(\varepsilon_*), \tag{5.17}$$

where Δ_0 *is given in* (5.11) *and* (5.12), *and* $\mu(\alpha,\alpha')$ *is bounded in* (5.16).

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